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Vibration Analysis of Rotor Blades with Pendulum Absorbers

V. R. Murthy*
Goodyear Aerospace Corporation, Akron, Ohio
and
C. E. Hammond†
U.S. Army Research and Technology Laboratories (AVRADCOM), Hampton, Va.

A comprehensive vibration analysis of rotor blades with spherical pendulum absorbers is presented. These absorbers are mounted on the blade to attenuate the loads that pass through the rotor shaft at a frequency that is an integer multiple of the blade passage frequency. The spherical pendulum absorber is considered here because such an absorber may be more versatile in controlling the out-of-plane, in-plane, and pitching vibratory loads. A transmission matrix formulation is given to determine the natural vibrational characteristics and a direct transmission matrix method is given to determine the forced response of the rotor blades with pendulum absorbers. The pendulum absorber alters significantly the initial blade frequencies that are adjacent to the uncoupled frequencies of the pendulum. In the case of spherical pendulum, two natural modes of vibration are injected between the displaced initial modes of the blade. The pendulum hinge offsets significantly affect the torsional frequencies of the blade. The effect of root collective pitch, in the normal range, on the natural frequencies is small. With the use of a spherical pendulum, it is possible to achieve significant reductions in all five root forces of the blade simultaneously. If the pendulum is tuned such that its uncoupled natural frequency coincides with the excitation frequency, the pendulum introduces constrained deflections at its point of connection with the blade; this may not help in attenuating the root forces of the blade but may actually amplify them.

Nomenclature

C_1, C_2, c_1	$= \cos\Theta_1$, $\cos\Theta_2$, $\cos\Theta_{01}$, respectively
L_y, L_z	= forces acting on the blade per unit span along y,z directions, respectively
l	= length of the pendulum
l_y, l_z	= amplitudes of simple harmonic forces L_y and L_z
M	= mass of the pendulum
M_x, M_y, M_z	= internal moments about x,y,z directions, respectively
m	= mass per unit span
m_x, m_y, m_z	= amplitudes of simple harmonic moments M_x, M_y, M_z, respectively
$[P]$	= point transmission matrix of the pendulum
Q	= twisting moment acting on the blade per unit span
q	= amplitude of simple harmonic twisting moment Q
R	= span of the blade
RS	= distance between the axis of rotation and the root of the blade
S_1, S_2, s_1	$= \sin\Theta_1$, $\sin\Theta_2$, $\sin\Theta_{01}$, respectively
T	= tension in the blade
$[T(x)]$	= transmission matrix of the blade
V, W	= in-plane and out-of-plane displacements, respectively
V_y, V_z	= shear forces along y,z directions, respectively
v, w	= amplitudes of simple harmonic displacements V, W, respectively
x,y,z	= Cartesian coordinate system that rotates with the blade; x axis falls along undeformed elastic axis, y axis is horizontal axis, and z axis is vertical axis
x_f	= spanwise location of the concentrated simple harmonic loading
$\{Z(x)\}$	= state vector containing the elements $W, V, \Psi, N, \Phi, M_x, M_z, M_y, -V_y, -V_z$ in that order
$\{z(x)\}$	= state vector containing the elements of the amplitudes of the simple harmonic quantities of $\{Z(x)\}$
β	= blade twist
η, ζ	= local cross-sectional coordinates; η is measured along chord, ζ is measured along a line perpendicular to the chord at the shear center; $y_A = \eta_A \cos\beta_A - \zeta_A \sin\beta_A$; $z_A = \eta_A \sin\beta_A + \zeta_A \cos\beta_A$
Θ_1, Θ_2	= generalized coordinates of the spherical pendulum
Θ_{01}, Θ_{02}	= steady-state deflections of the spherical pendulum
θ_1, θ_2	= deflections of the pendulum; $\Theta_1 = \Theta_{01} + \theta_1$; $\Theta_2 = \Theta_{02} + \theta_2$
θ_1^*, θ_2^*	= amplitudes of simple harmonic pendulum deflections θ_1, θ_2
N, Ψ	$= \mathrm{d}V/\mathrm{d}x$, $\mathrm{d}W/\mathrm{d}x$, respectively
ν, ψ, ϕ	= amplitudes of N, Ψ, Φ, respectively
Φ	= torsional deflection
Ω	= rotational speed
ω	= frequency of vibration
ω_{p1}, ω_{p2}	= uncoupled natural frequencies of the pendulum

Subscripts

A, p	= at the pendulum hinge and mass locations, respectively
1	= deformed rotating coordinates

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*Engineer Specialist. Member AIAA.
†Leader, Rotorcraft Aeroelasticity, Structures Laboratory. Member AIAA.

Superscripts

L, R	= immediate left and right, respectively
T	= transpose of the matrix
-1	= inverse of the matrix
$(')$	= differentiation with respect to x
$(\dot{\ })$	= differentiation with respect to time

Introduction

HELICOPTERS operate in a severe vibrational environment, and it is important to maintain a low level of vibration for the comfort of crew and passengers, to minimize maintenance problems, and to increase the fatigue life of the blades and other structural components. The total design technology for control of helicopter vibrations consists mainly of three basic approaches: 1) reduction of vibratory loads at the sources, 2) modification of the response dynamics of the system, and 3) isolation of the vibration.

The rotor system that transmits the vibratory loads to the fuselage through the rotor shaft is one of the most significant contributors to the helicopter vibration. Since the vibratory airloads on the rotor are functions of the blade dynamic characteristics, it is possible to achieve reductions in the total aircraft vibration levels by suitable modifications of the rotor dynamics. Blade-mounted pendulum absorbers modify the dynamics of the blades, and, through the use of these devices, significant reductions in the helicopter vibration levels have been reported.[1-4]

In this paper a comprehensive vibration analysis of rotor blades with spherical pendulum absorbers is presented. The nonlinear equations of motion of the spherical pendulum on elastic rotor blades undergoing coupled flapwise bending, chordwise bending, and torsional vibrations are derived. The nonlinear equations of motion are linearized for small oscillations about the steady-state deflections of the pendulum, and the linearized equations are combined with the linear blade equations[5] by means of transmission matrices. The spherical pendulum absorber is considered here since it is expected that such an absorber may be more versatile in controlling the out-of-plane, in-plane, and pitching vibratory loads. A transmission matrix formulation is presented to determine the natural vibrational characteristics of rotor blades with pendulum absorbers. A direct transmission matrix method is given to determine the forced response of the rotor blades with the pendulum absorbers. The transmission matrix of the basic blade with coupled flapwise bending, chordwise bending, and torsional degree of freedom, by considering the continuous model, is obtained by the approach used in Ref. 6. The formulations are appealing because of their simplicity in programming for digital computer calculations. The inputs are generated very easily since they are merely the coefficients of the basic blade differential equations and the elements of the transmission matrix of the pendulum on the rotor blade.

The natural vibrational characteristics of a hingeless rotor blade with a spherical pendulum are computed, and these results are compared with the initial blade characteristics and with those obtained by using a simple pendulum or a concentrated mass. The effects of spanwise location of the pendulum, hinge offset of the pendulum, and root collective pitch of the blade on the natural frequencies are determined. The attachment of a pendulum on a rotor blade alters significantly the initial frequencies of the blade adjacent to the uncoupled natural frequencies of the pendulum. New natural modes of vibration corresponding to the degrees of freedom of the pendulum are injected between the displaced initial frequencies of the blade.

In general, a properly tuned pendulum attenuates the vibratory loads in two ways: 1) by eliminating the resonant or nearly resonant responses of the blade by displacing the initial blade natural frequencies that are in the vicinity of excitation frequency, and 2) by generating appropriate forces at its point of connection with the blade. These forces redistribute the loads on the blade so that the forces at the root of the blade are attenuated. If the pendulum absorber is tuned so that its uncoupled natural frequency coincides with the excitation frequency, the pendulum introduces constrained deflections at its point of connection with the blade. This may not help in attenuating the root forces of the blade and may actually amplify them. The root simple harmonic responses of a hingeless rotor blade with a spherical pendulum absorber and with a simple pendulum absorber are computed, and these results are compared with the responses of the blade without the pendulum. It is found that it is possible to achieve significant reductions in all five forces of the blade simultaneously with the use of a spherical pendulum absorber.

Pendulum Equations

A. Nonlinear Equations of Motion

The blade-mounted spherical pendulum absorber, together with the coordinate system, is shown in Fig. 1. The rotating deformed coordinates of the pendulum hinge and mass can be related from Fig. 2 as

$$x_{1p} = x_{1A} + lS_2C_1; \qquad y_{1p} = y_{1A} - lS_2C_1; \qquad z_{1p} = z_{1A} - lC_2$$

The inertial accelerations of the spherical pendulum on the rotor blade can be obtained by substituting the above equations in Eq. (B7) of Ref. 5. These, after neglecting the terms associated with axial degree of freedom, are given by

$$\begin{aligned} A_{xp} = & -y_A \ddot{V}'_A - z_A \ddot{W}'_A + \Omega^2 (y_A V'_A + z_A W'_A) - 2\Omega(\dot{V}_A - z_A \dot{\Phi}_A) \\ & - \Omega^2 (x_A + lS_2C_1) + l(-S_1S_2\ddot{\theta}_1 + C_1C_2\ddot{\theta}_2) \\ & + 2\Omega l(S_2C_1\dot{\theta}_1 + S_1C_2\dot{\theta}_2) - l\{S_2C_1(\dot{\theta}_1^2 + \dot{\theta}_2^2) + 2S_1C_2\dot{\theta}_1\dot{\theta}_2\} \end{aligned} \tag{1a}$$

$$\begin{aligned} A_{yp} = & -2\Omega(y_A \dot{V}'_A + z_A \dot{W}'_A) + \ddot{V}_A - z_A\ddot{\Phi}_A - \Omega^2 (V_A + y_A \\ & - z_A\Phi_A - lS_1S_2) - l(S_2C_1\ddot{\theta}_1 + S_1C_2\ddot{\theta}_2) + 2\Omega l(-S_1S_2\dot{\theta}_1 \\ & + C_1C_2\dot{\theta}_2) - l\{-S_1S_2(\dot{\theta}_1^2 + \dot{\theta}_2^2) + 2C_1C_2\dot{\theta}_1\dot{\theta}_2\} \end{aligned} \tag{1b}$$

$$A_{zp} = \ddot{W}_A + y_A\ddot{\Phi}_A + lS_2\ddot{\theta}_2 + lC_2\dot{\theta}_2^2 \tag{1c}$$

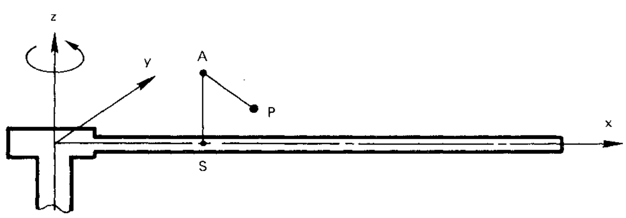

Fig. 1 Blade-mounted pendulum absorber before deformation.

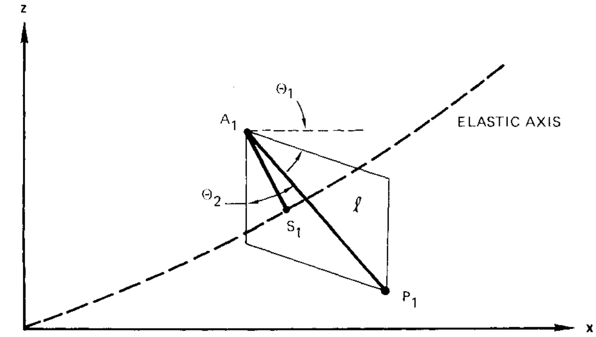

Fig. 2 Pendulum location after deformation.

The inertial forces due to the above accelerations should produce zero moments about the pendulum hinge point (point A_1 of Fig. 2) and this requirement yields the following nonlinear equations:

$$lS_2\ddot{\Theta}_1 - 2\Omega lC_2\dot{\Theta}_2 + 2lC_2\dot{\Theta}_1\dot{\Theta}_2 + \Omega^2 C_1 V_A + 2\Omega S_1 \dot{V}_A - C_1\ddot{V}_A - \Omega^2 S_1 Y_A V'_A - 2\Omega C_1 y_A \dot{V}'_A + S_1 y_A \ddot{V}'_A - \Omega^2 S_1 z_A W'_A - 2\Omega C_1 z_A \dot{W}'_A + S_1 z_A \ddot{W}'_A - \Omega^2 C_1 Z_A \Phi_A - 2\Omega S_1 z_A \dot{\Phi}_A + C_1 z_A \ddot{\Phi}_A + \Omega^2 (S_1 x_A + C_1 y_A) = 0 \tag{2}$$

$$l\ddot{\Theta}_2 - lS_2C_2\dot{\Theta}_1^2 - \Omega^2(x_A C_1 + lS_2 - y_A S_1)C_2 + 2\Omega lS_2C_2\dot{\Theta}_1 + S_2(\ddot{W}_A + y_A\ddot{\Phi}_A) + S_1C_2[\Omega^2(V_A - z_A\Phi_A) - (\ddot{V}_A - z_A\ddot{\Phi}_A)] - 2\Omega S_1 C_2(y_A\dot{V}'_A + z_A\dot{W}'_A) - 2\Omega C_2C_1(\dot{V}_A - z_A\dot{\Phi}_A) + \Omega^2 C_2 C_1(y_A V'_A + z_A W'_A) - C_2C_1(\ddot{V}'_A + \ddot{W}'_A) = 0 \tag{3}$$

B. Linearized Equations of Motion

The nonlinear equations of motion given by Eqs. (2) and (3) are linearized for small oscillations about the steady-state equilibrium position of the pendulum. The steady-state equilibrium angles (Θ_{01}, Θ_{02}) are obtained from Eqs. (2) and (3) by treating the acceleration and velocity terms as zero, and this process yields

$$\Omega^2\cos\Theta_{01}(V_A - z_A\Phi_A) - \Omega^2\sin\Theta_{01}(y_A V'_A + z_A W'_A) + \Omega^2(x_A\sin\Theta_{01} + y_A\cos\Theta_{01}) = 0 \tag{4a}$$

$$\Theta_{02} = 90 \text{ deg} \tag{4b}$$

By substituting $\Theta_1 = \Theta_{01} + \theta_1, \Theta_2 = \Theta_{02} + \theta_2$ in Eqs. (2) and (3) and neglecting the nonlinear terms in Θ_1 and Θ_2, the following linearized equations of motion for the spherical pendulum are obtained:

$$l\ddot{\theta}_1 + \Omega^2 c_1 V_A + 2\Omega s_1\dot{V}_A - c_1\ddot{V}_A - \Omega^2 s_1 y_A V'_A - 2\Omega c_1 y_A \dot{V}'_A + y_A s_1 \ddot{V}'_A - \Omega^2 s_1 z_A W'_A - 2\Omega c_1 z_A \dot{W}'_A + s_1 z_A \ddot{W}'_A - \Omega^2 c_1 z_A \Phi_A - 2\Omega s_1 z_A \dot{\Phi}_A + c_1 z_A \ddot{\Phi}_A + \Omega^2(c_1 x_A - s_1 y_A)\theta_1 = 0 \tag{5}$$

$$l\ddot{\theta}_2 + \Omega^2\theta_2(x_A c_1 + l - y_A s_1) + \ddot{W}_A + y_A\ddot{\Phi}_A = 0 \tag{6}$$

For the linear analysis of the blade plus pendulum Eq. (4a) becomes

$$x_A\sin\Theta_{01} + y_A\cos\Theta_{01} = 0 \tag{7}$$

C. Uncoupled Pendulum Natural Frequencies

The linearized equations of motion of the spherical pendulum on a rigid rotor blade can be obtained from Eqs. (5) and (6) by treating $W_A = V_A = \Phi_A = 0$. The resulting equations yield the following two natural frequencies for the spherical pendulum:

$$\omega_{p1} = \Omega[(x_A\cos\Theta_{01} - y_A\sin\Theta_{01})/l]^{1/2} \tag{8a}$$

$$\omega_{p2} = \Omega[(x_A\cos\Theta_{01} + l - y_A\sin\Theta_{01})/l]^{1/2} \tag{8b}$$

D. Pendulum Deflections in Coupled Mode Shapes

By seeking simple harmonic solutions of the form $\theta_1 = \theta_1^*\exp(i\omega t)$, $W_A = w_A\exp(i\omega t)$, $V_A = v_A\exp(i\omega t)$, and $\Phi_A = \phi\exp(i\omega t)$ for Eqs. (5) and (6), the following equations for the pendulum deflections can be obtained:

$$\theta_1^* = (\omega^2 + \Omega^2)[c_1(v_A - z_A\phi_A) - s_1(y_A v'_A + z_A w'_A)]/a_1 + i2\omega\Omega[s_1(v_A - z_A - \phi_A) - c_1(y_A v'_A + z_A w'_A)]/a_1 \tag{9a}$$

$$\theta_2^* = \omega^2(w_A + y_A\phi_A)/a_2 \tag{9b}$$

where

$$a_1 = \omega^2 l - \Omega^2(c_1 x_A - s_1 y_A); \qquad a_2 = \Omega^2 l - a_1$$

E. Pendulum Transmission Matrix

The linearization of inertial accelerations given by Eq. (1) yields the following linear accelerations:

$$A_{xp} = y_A\ddot{V}'_A + z_A\ddot{W}'_A - \Omega^2(y_A V'_A + z_A W'_A) + 2\Omega(\dot{V}_A - z_A\dot{\Phi}_A) + \Omega^2(x_A + lc_1 - ls_1\theta_1) + ls_1\ddot{\theta}_1 - 2\Omega lc_1\dot{\theta}_1 \tag{10a}$$

$$A_{yp} = 2\Omega(y_A\dot{V}'_A + z_A\dot{W}'_A) - (\ddot{V}_A - z_A\ddot{\Phi}_A) + \Omega^2(V_A + y_A - z_A\Phi_A) - \Omega^2 l(s_1 + c_1\theta_1) + lc_1\ddot{\theta}_1 + 2\Omega ls_1\dot{\theta}_1 \tag{10b}$$

$$A_{zp} = -(\ddot{W}_A + y_A\ddot{\Phi}_A + l\ddot{\theta}_2) \tag{10c}$$

The moment vector of the inertial forces about the deformed position of the shear center of the blade cross section, at which the pendulum is connected (point S_1 in Fig. 2) is given by

$$M = rxF = iM_{xp} + jM_{yp} + kM_{zp} \tag{11}$$

where (referring to Fig. 2)

$$r = S_1A_1 = -i(y_A V'_A + z_A W'_A) + j(y_A - z_A\Phi_A) + k(z_A + y_A\Phi_A)$$

$$F = M(iA_{xp} + jA_{yp} + kA_{zp}) = iF_{xp} + jF_{yp} + kF_{zp} \tag{12}$$

where A_{xp}, A_{yp} and A_{zp} are defined in Eq. (10).

The equilibrium equations across the pendulum are given by (see Fig. 3)

$$T^R = T^L - F_{xp} \tag{13}$$

$$M_x^R = M_x^L - M_{xp} \qquad M_z^R = M_z^L - M_{zp} \qquad M_y^R = M_y^L + M_{yp}$$
$$-V_y^R = -V_y^L + F_{yp} \qquad -V_z^R = -V_z^L + F_{zp} \tag{14}$$

where F_{xp}, F_{yp}, F_{zp}, M_{xp}, M_{yp} and M_{zp} are defined in Eqs. (11) and (12).

The pendulum transmission matrix is now obtained by the following steps: 1) assuming simple harmonic solutions of the form

$$\{Z\} = \{z\}\exp(i\omega t), \qquad \theta_1 = \theta_1^*\exp(i\omega t), \qquad \theta_2 = \theta_2^*\exp(i\omega t)$$

for Eq. (14) where

$$\{Z\}^T = [W, V, \Psi, N, \Phi, M_x, M_z, M_y, -V_y, -V_z]$$

$$\{z\}^T = [w, v, \psi, \nu, \phi, m_x, m_z, m_y, -v_y, -v_z]$$

2) substituting Eqs. (9a) and (9b) in the equations obtained by step 1, 3) cancelling the exponential time dependency in the resulting equations, and 4) arranging the equations obtained

after step 3 and the equations $w^R = w^L; v^R = v^L; \psi^R = \psi^L; \nu^R = \nu^L; \phi^R = \phi^L$ into a matrix equation of the following form:

$$\{z\}^R = [P]\{z\}^L \tag{15}$$

The matrix $[P]$ in Eq. (15) defines the transmission matrix of the spherical pendulum on the rotor blade. The nonzero elements of the matrix are given in the following:

$$P_{11} = P_{22} = P_{33} = P_{44} = P_{55} = P_{66} = P_{77} = P_{88} = P_{99} = P_{1010} = 1$$

$$P_{61} = -M\omega^2 y_A (1 + \omega^2 l/a_2); \qquad P_{62} = Mz_A d_1$$

$$P_{63} = Mz_A^2 d_2; \qquad P_{64} = My_A zAd_2$$

$$P_{65} = y_A P_{61} - z_A P_{62} - M\Omega^2 y_A (ls_1 - y_A)$$

$$P_{72} = My_A d_3; \qquad P_{73} = Mz_A d_4; \qquad P_{74} = My_A d_4$$

$$P_{75} = -z_A P_{72} - M\Omega^2 z_A (x_A + lc_1); \qquad P_{82} = Mz_A d_3$$

$$P_{83} = Mz_A d_5; \qquad P_{84} = My_A d_5$$

$$P_{85} = -z_A P_{82} + M\Omega^2 y_A (x_A + lc_1)$$

$$P_{92} = Md_1; \qquad P_{93} = Mz_A d_2$$

$$P_{94} = My_A d_2; \qquad P_{95} = -z_A P_{92}$$

$$P_{101} = M\omega^2 (1 + \omega^2 l/a_2); \qquad P_{105} = y_A P_{101}$$

$$d_1 = (\omega^2 + \Omega^2) - (l/a_1)\{(\omega^2 + \Omega^2)^2 c_1^2 + 4\omega^2\Omega^2 s_1^2\}$$

$$d_2 = lc_1 s_1 \{(\omega^2 + \Omega^2)^2 + 4\omega^2\Omega^2\}/a_1 + i2\omega\Omega\{1 + (\omega^2 + \Omega^2)(c_1^2 - s_1^2) l/a_1\}$$

$$d_3 = -(\omega^2 - \Omega^2)^2 lc_1 s_1/a_1 - i2\omega\Omega\{(\omega^2 + \Omega^2) l/a_1 - 1\}$$

$$\begin{Bmatrix} w(x) \\ v(x) \\ \phi(x) \end{Bmatrix} = \begin{bmatrix} T_{16}(x) & T_{17}(x) & T_{18}(x) & T_{19}(x) \\ T_{26}(x) & T_{27}(x) & T_{28}(x) & T_{29}(x) \\ T_{56}(x) & T_{57}(x) & T_{58}(x) & T_{59}(x) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} \begin{Bmatrix} T_{110}(x) \\ T_{210}(x) \\ T_{310}(x) \end{Bmatrix} \tag{22}$$

$$d_4 = -(\omega^2 y_A + \Omega^2 ls_1) + y_A l\{(\omega^2 + \Omega^2)^2 s_1^2 - 4\omega^2\Omega^2 c_1^2\}/a_1 + i4\omega\Omega y_A lc_1 s_1 (\omega^2 + \Omega^2)/a_1$$

$$d_5 = -(\omega^2 + \Omega^2) z_A + z_A l\{(\omega^2 + \Omega^2)^2 s_1^2 - 4\omega^2\Omega^2 c_1^2\} + i4\omega\Omega(\omega^2 + \Omega^2) z_A lc_1 s_1/a_1$$

Transmission Matrix of the Blade Plus Pendulum

The transmission matrix of the blade without a pendulum is given by[6]

$$[T(x)]' = [A(x)][T(x)] \tag{16}$$

$$[T(0)] = (I) \tag{17}$$

where $[A(x)]$ is defined in the Appendix. The tension coefficient T appearing in matrix $[A]$ is given by

$$T = \Omega^2 \int_x^R mx_1 \mathrm{d}x_1 \tag{18}$$

When a pendulum is attached to the blade the transmission matrix at any spanwise location is obtained as follows. Let x_A be the spanwise location of the pendulum.

A. Case 1 $0 \le x \le x_A^L$

The transmission matrix in this case is obtained by simply integrating Eq. (16) together with the initial conditions given by Eq. (17) similar to the case of the blade without the pendulum. However, the tension coefficient given by the following equation should be used instead of the one given by Eq. (18)

$$T = \Omega^2 \int_x^R mx_1 \mathrm{d}x_1 + M\Omega^2 (x_A + lc_1) \tag{19}$$

B. Case 2 $x_A^R \le x \le R$

Let $[T_1]$ be the transmission matrix of the blade up to $x = x_A^L$; this matrix can be obtained from case 1 as just discussed. Let $[P]$ be the point transmission matrix of the pendulum. Let $[T_2(x)]$ be the transmission matrix of the blade from x_A^R to x. This matrix is obtained by the integration of Eqs. (16) and (17) from x_A to $(x - x_A)$. While integrating this equation the tension coefficient T given by Eq. (18) should be used instead of the one given by Eq. (19). By the product rule of the transmission matrices,[7] the transmission matrix of the system for this case is given by

$$[T(x)] = [T_2(x)][P][T_1] \tag{20}$$

Natural Frequencies and Mode Shapes

Once the system transmission matrix has been determined, the natural frequencies and the associated modal functions can be determined as discussed in Ref. 6. The frequency determinant and the modal equation corresponding to clamped-free boundary conditions of the rotor blade are given by

$$\begin{vmatrix} T_{66} & T_{67} & T_{68} & T_{69} & T_{610} \\ T_{76} & T_{77} & T_{78} & T_{79} & T_{710} \\ \vdots & & & & \\ T_{106} & T_{107} & T_{108} & T_{109} & T_{1010} \end{vmatrix} = 0 \tag{21}$$

where

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} T_{66} & T_{67} & T_{68} & T_{69} \\ T_{76} & T_{77} & T_{78} & T_{79} \\ T_{86} & T_{87} & T_{88} & T_{89} \\ T_{96} & T_{97} & T_{98} & T_{99} \end{bmatrix}^{-1} \begin{Bmatrix} T_{610} \\ T_{710} \\ T_{810} \\ T_{910} \end{Bmatrix}$$

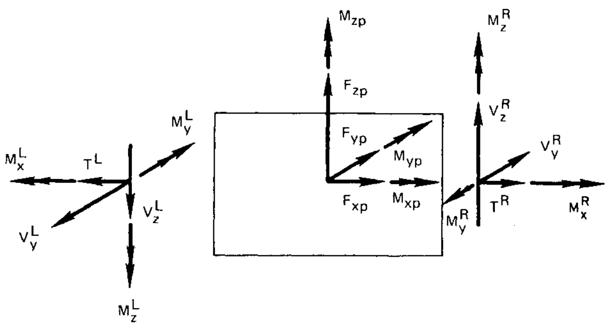

Fig. 3 Free-body diagram of blade element containing pendulum.

Dynamic Response

The Fourier transforms of basic differential equations of motion of rotor blades can be arranged into a matrix differential equation of the following form:

$$\{z(x)\}' = [A(x)]\{z(x)\} + \{f(x)\} \tag{23}$$

where $A(x)$ is defined under Eq. (16), $\{z(x)\}$ is the Fourier transform of state vector $\{Z(x)\}$, and $\{f(x)\}$ is the Fourier transform of the loading vector. The nonzero elements of $\{f(x)\}$ are $f_6(x) = -q(x)$, $f_9(x) = l_y(x)$ and $f_{10}(x) = l_z(x)$.

The complete solution of Eq. (23) can be written as[8]

$$\{z(x)\} = [T(x)]\{z(0)\} + [T(x)]\int_0^x [T(x_1)]^{-1}\{f(x_1)\}dx_1 \tag{24}$$

In principle, the solution vector of the problem at hand viz., the response vector due to loadings Q, L_y, and L_z, is given by the inverse Fourier transform of Eq. (24). But if the loadings Q, L_y, and L_z are simple harmonic with frequency ω, then Eq. (24) directly gives the amplitude vector of the response. The transmission matrix of Eq. (24) is obtained as discussed before and the vector $\{z(0)\}$ is not known completely at the present time. The determination of this vector is shown in the following analysis of the blade subjected to simple harmonic forces.

If the blade is subjected to the concentrated simple harmonic forces at x_f, the force vector $\{f(x)\}$ can be expressed as

$$\{f(x)\} = \delta(x - x_f)\{f\} \tag{25}$$

substituting Eq. (25) into Eq. (24) yields

$$\left\{\begin{array}{c} z_d(x) \\ \hline z_f(x) \end{array}\right\} = \left[\begin{array}{c|c} T_a(x) & T_b(x) \\ \hline T_c(x) & T_d(x) \end{array}\right] \left\{\begin{array}{c} z_d(0) \\ \hline z_f(0) \end{array}\right\} \quad ; \quad x < x_f \tag{26}$$

$$\left\{\begin{array}{c} z_d(x) \\ \hline z_f(x) \end{array}\right\} = \left[\begin{array}{c|c} T_a(x) & T_b(x) \\ \hline T_c(x) & T_d(x) \end{array}\right] \left(\left\{\begin{array}{c} z_d(0) \\ \hline z_f(0) \end{array}\right\} + \left[\begin{array}{c|c} S_a(x_f) & S_b(x_f) \\ \hline S_c(x_f) & S_d(x_f) \end{array}\right] \left\{\begin{array}{c} f_d \\ \hline f_f \end{array}\right\} \right) \quad x \geq x_f \tag{27}$$

where

$$\{z_d\}^T = [w, v, \psi, \nu, \phi]; \qquad \{z_f\}^T = [m_x, m_z, m_y, -v_y, -v_z]$$

$$\{f_d\} = \{0\}; \qquad \{f_f\}^T = [-q, 0, 0, l_y, l_z]$$

$$[S(x_f)] = [T(x_f)]^{-1}$$

The boundary conditions corresponding to a hingeless blade are $\{z_d(0)\} = \{0\}$ and $\{z_f(R)\} = 0$. The substitution of these conditions into Eqs. (26) and (27) yields

$$\{z_f(0)\} = -[T_d(R)]^{-1}([T_c(R)][S_b(x_f)] + [T_d(R)][S_d(x_f)])\{f_f\} \tag{28}$$

Once the transmission matrix of the system is known, the response vector of the system subjected to the simple harmonic forces $\{f_f\}$ can be determined from Eqs. (26) and (27) by making use of the boundary conditions and Eq. (28). Although Eqs. (26) and (27) are applicable for concentrated simple harmonic forces, the procedure is applicable for distributed simple harmonic forces in which case the numerical evaluation of Eq. (24) may be required. In the case of nonsimple harmonic loadings, the response analysis by using the normal modes may be preferred to avoid taking of the inverse Fourier transform of Eq. (24). If the system is subjected to the oscillatory forces with a frequency which is equal to the uncoupled natural frequency of the pendulum, then the method just stated breaks down since the pendulum transmission matrix is singular at this frequency. In this case the blade deflections are constrained at the blade attachment point as will be shown.

$$\text{Case 1:} \quad \omega = \omega_{p1} \qquad v_A = z_A \phi_A$$

$$\text{Case 2:} \quad \omega = \omega_{p2} \qquad w_A = -y_A \phi_A \tag{29}$$

In this case the pendulum transmission matrix has to be modified according to constraint given in Eq. (29). This constraint introduces an unknown reaction at the pendulum attachment point and a procedure for the evaluation of this reaction is given in Ref. 8.

Numerical Results and Discussion

The numerical results presented here pertain to a uniform hingeless rotor blade undergoing coupled flapwise bending, chordwise bending, and torsional vibrations with the following properties:

$$R = 260\text{ in.} = 6.6\text{ m}; \quad \beta = -10\text{ deg}$$

$$mg = 0.5796\text{ lb/in.} = 101.51\text{ N/m}$$

$$EI_1 = 0.3 \times 10^8\text{ lb-in.}^2 = 0.86 \times 10^5\text{ Nm}^2$$

$$EI_2 = 0.1 \times 10^{10}\text{ lb-in.}^2 = 28.69 \times 10^5\text{ Nm}^2$$

$$GJ = 0.2 \times 10^8\text{ lb-in.}^2 = 0.57 \times 10^5\text{ Nm}^2$$

$$mgk_{m_1}^2 = 0.35\text{ lb-in.} = 0.04\text{ Nm}$$

$$mgk_{m_2}^2 = 15.456\text{ lb-in.} = 1.7464\text{ Nm}$$

$$RS = 0; \qquad e = -0.6\text{ in.} = -1.524\text{ cm}$$

$$M = 15\text{ lb} = 66.726\text{ N}; \qquad l = 6.0\text{ in.} = 15.24\text{ cm}$$

A. Natural Frequencies and Mode Shapes

The natural frequencies of the rotor blade for the cases: 1) with a spherical pendulum, 2) with a simple pendulum, 3) with a concentrated mass, and 4) with none of the above are presented in Table 1 together with the pendulum deflections. The transmission matrices for simple pendulum and for concentrated mass are given in Ref. 8 and these can be obtained by suitably specializing the results of the spherical pendulum.

It can be observed from Table 1 that two additional frequencies appear between the displaced modes in the case of the spherical pendulum, and one additional mode appears in the case of the simple pendulum. The appearance of the additional modes (corresponding to the pendulum degrees of freedom) between the displaced blade modes is indicated mathematically by the singular behavior of the frequency equations as shown in Fig. 4. This singular behavior occurs at the uncoupled natural frequencies of the pendulum. When only a concentrated mass is added to the blade there is no singular behavior in the frequency equation, as can be seen from Fig. 4. The pendulum affects original blade modes more significantly than a mere concentrated mass.

The mode shapes associated with the blade natural frequencies presented in Table 1 are computed, and the following observations are made.

Table 1 Comparison of natural frequencies ($\Omega = 360$ rpm; $X_A = 1.65$ m; $\eta_A = 0$; $\zeta_A = 0$; collective pitch = 0 deg)

Mode	Blade β,rad/s	With mass ω,rad/s	With simple pendulum[a] ω,rad/s	With simple pendulum[a] θ_2, deg	With spherical pendulum[b] ω, rad/s	With spherical pendulum[b] θ_1, deg	With spherical pendulum[b] θ_2, deg
1	40.04 F[c]	40.03 F	40.03 F	2.04	40.03 F	−0.15	2.04
2	45.31 C[d]	45.22 C	45.21 C	−0.31	45.19 C	−2.93	−0.34
3	105.74 F	102.31 F	97.27 F	−55.69	97.25 F	4.19	−55.65
PM					122.61 C	287.37	66.24
PM			131.24 F	154.96	131.28 F	−14.31	154.27
4	138.80 T[e]	138.61 T	139.81 T	−35.56	139.82 T	3.10	−35.77
5	201.76 F	186.18 F	214.26 F	−106.85	214.33 F	−4.41	−106.22
6	281.94 C	271.18 C	270.26 C	9.12	285.06 C	−56.80	6.69
7	336.90 F	318.51 F	342.67 F	82.57	342.70 F	−2.58	82.91
8	402.73 T	402.31 T	402.83 T	8.41	402.84 T	−0.83	8.44

[a] $\omega_{p2} = 129.68$ rad/s, $\Theta_{02} = 90$ deg. [b] $\omega_{p1} = 124.08$ rad/s, $\omega_{p2} = 129.68$ rad/s, $\Theta_{01} = 0$, $\Theta_{02} = 90$ deg. [c] F = predominantly flapwise bending. [d] C = predominantly chordwise bending. [e] T = predominantly torsion.

Table 2 Comparison of root forces ($\Omega = 360$ rpm; $\eta_A = 0$; $\zeta_A = 20.32$ cm; collective pitch = 12 deg; $M = 66.73$ N; $\omega = 150.7965$ rad/s; $q = l_y = 2224$ N; $x_f = 6.6$ m)

Case	m_x, Nm	m_z, Nm	m_y, Nm	v_y, N	v_z, N
Without pendulum	−139	−648	1536	−1054	3256
Simple pendulum					
$l = 12.15$ cm					
$x_A = 1.6510$ m	−5	−576	58	−983	−0.5
Spherical pendulum					
$l = 12.70$ cm					
$x_A = 1.3970$ m	130	67	936	−298	1762

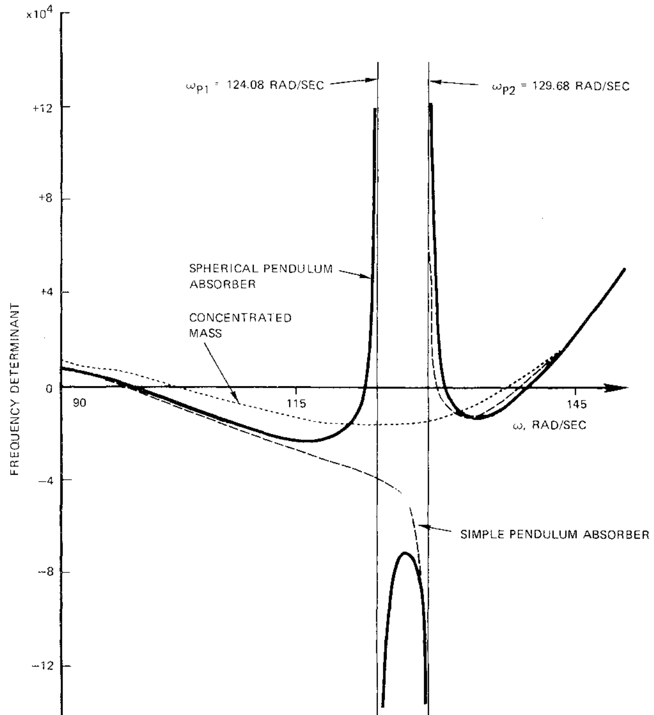

Fig. 4 Frequency determinant plot.

1) The attachment of a spherical pendulum on a rotor blade replaces two of the original blade frequencies by four frequencies; two of these can be thought of as displaced original frequencies, and the other two as the coupled pendulum frequencies. It is observed that the modes corresponding to the displaced original frequencies differ slightly from the original modes whereas the pendulum modes differ significantly from them. In Table 1, the coupled natural frequencies of the pendulum are denoted by PM in the column of mode numbers.

2) The maximum pendulum deflections are observed to occur usually in the pendulum modes.

3) In the case of the spherical pendulum, one pendulum mode is predominantly chordwise bending mode and the other is predominantly flapwise bending mode. In the case of the simple pendulum, the mode is predominantly flapwise bending mode.

4) In the case of the spherical pendulum, the original frequencies that are below the uncoupled pendulum frequencies are lowered and the original frequencies that are above the uncoupled pendulum frequencies are raised. In the case of the simple pendulum a similar behavior is observed with the exception of the predominantly chordwise bending frequencies.

In addition to the uncoupled pendulum frequencies, the spanwise location of the pendulum plays an important role in displacing the initial frequencies of the blade since the deflection at the pendulum attachment point actually couples the pendulum with the blade. The results obtained by varying the pendulum hinge vertical offset (ζ_A) indicate that this parameter alters the torsional frequencies significantly. The effect of the collective pitch (between 0 and 16 deg) on the natural frequencies is found to be small and its effect with the pendulum is much the same as its effect without the pendulum. However, at higher collective pitch angles the effect could be significant because of the changes in y_A and z_A coordinates (η_A and ζ_A are fixed) associated with the collective pitch change.

B. Dynamic Response

The undamped root simple harmonic responses of the blade with a properly tuned spherical pendulum and a simple pendulum are computed and these are presented in Table 2 together with the results obtained without the pendulum. The excitation force for these results is a concentrated out-of-plane simple harmonic load at the tip of the blade with a frequency equal to 4/rev. The proper tunings of the pendulums are determined by comparing the results obtained by varying the pendulum parameters: 1) spanwise location, 2) length, and 3) mass. The results presented in Table 2 indicate that by a suitable choice of spherical pendulum parameters, it is possible to attenuate significantly all five root forces of the blade. In the case of simple pendulum, it is possible to attenuate out-of-plane and twisting forces.

In general, a properly tuned pendulum can attenuate the vibratory loads in two ways: 1) by eliminating the resonant responses of the blade by displacing the initial frequencies that are in the vicinity of the excitation frequency, and 2) by generating appropriate forces at its point of connection with the blade. These forces redistribute the loads on the blade so that the forces at the root of the blade are attenuated. If the pendulum absorber is tuned so that its uncoupled natural frequency coincides with the excitation frequency, the pendulum constrains the deflections at its point of connection with the blade; this may not help in attenuating the root forces of the blade and may actually amplify them.

The root forces of the blade, by varying the spanwise location of the applied concentrated force, are computed. These results are compared with those obtained without the pendulum. From these results the following observations are made.

1) If the simple harmonic force is acting at the tip of the blade the pendulum attenuates the root forces significantly (actually the pendulum was tuned like this).

2) If the load is applied at the other spanwise locations, the root forces are attenuated significantly to a lesser or more extent depending on the location.

The preceding observations indicate that even if a distributed simple harmonic load is applied to the blade, the net effect would be the attenuation of the root forces. Thus, it can be generally expected that a pendulum tuned for a concentrated simple harmonic load acting at the tip of the blade can very well serve the purpose for any distributed simple harmonic loading of the same frequency.

Appendix: Coefficient Matrix of the Blade Equations

The nonzero elements of the matrix $A(x)$ in Eq. (16) are now given (nomenclature as cited in Ref. 5).

$$A_{13} = A_{24} = A_{79} = A_{810} = 1$$

$$A_{37} = -(EI_2 - EI_1)\sin\beta\cos\beta / D$$

$$A_{38} = (EI_1 \sin^2\beta + EI_2 \cos^2\beta)/D$$

$$A_{47} = (EI_1 \cos^2\beta EI_2 \sin^2\beta)/D$$

$$A_{48} = A_{37} \qquad A_{56} = 1/GJ \qquad A_{61} = -\omega^2 me\cos\beta$$

$$A_{62} = (\omega^2 + \Omega^2) me\sin\beta$$

$$A_{63} = \Omega^2 mxe\cos\beta \qquad A_{64} = -\Omega^2 mxe\sin\beta$$

$$A_{65} = \Omega^2 m(k_{m_2}^2 - k_{m_1}^2)\cos 2\beta - \omega^2 mk_m^2$$

$$A_{74} = T \qquad A_{75} = A_{64} \qquad A_{83} = A_{74} \qquad A_{85} = A_{63}$$

$$A_{92} = (\omega^2 + \Omega^2) m \qquad A_{95} = -A_{62} \qquad A_{101} = \omega^2 m$$

$$A_{105} = -A_{61}$$

where

$$D = (EI_1 \cos^2\beta + EI_2 \sin^2\beta)(EI_1 \sin^2\beta + EI_2 \times \cos^2\beta) - (EI_2 - EI_1)^2 \cos^2\beta \sin^2\beta$$

$$T = \int_x^R \Omega^2 mx_1 \mathrm{d}x_1$$

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